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ELEMENTS

OF

PURE ARITHMETIC,

OR

NUMERICAL OPERATIONS

AND

THEIR PRIMARY RELATIONSHIPS

VIEWED AS THEY ARE IN THEMSELVES
WITHOUT REGARD TO NOTATION OR SYMBOLS.

BY

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> "Puck. Yet but three? Come one more;
> Two of both kinds makes up four."

LONDON:
LONGMAN, BROWN, GREEN, LONGMANS, AND ROBERTS.
MANCHESTER: THOS. SOWLER AND SONS.
1859.

181. c. 46.

BIBLIOTHECA · BODLEIANA

PREFACE.

THE aim of this little book is to lay a sure foundation for notational and symbolic arithmetic by setting forth shortly but thoroughly the essential nature and fundamental properties of numerical operations wholly disentangled from the consideration of any particular method of denoting numbers.

The unceasing call for easy modes of calculation in the daily business of life, along with the tendency to pass over without examination things so familiar and such mere matters of course as the simpler relations of numbers, has led to the custom of looking at arithmetic only through the medium of the ordinary system of notation and hence of mixing up together undistinguishably numerical operations and notational processes, until at last the foundation of pure science whereon the whole rests has been well nigh lost sight of and arithmetic and cyphering have become in fact almost convertible terms. Although doubtless arithmetical results of much value can be gained, at least without a sheer waste of labour, only by means of numerical notation and notational processes, yet it is precisely the nature and properties of numerical operations strictly so called out of which that very notation and those very processes spring, without which they would not be, and by understanding which consequently they can alone be understood. The readiest, if indeed not the only, way to arrive at a true knowledge of arithmetic, and much more of its mighty offshoot algebra, seems therefore to be to seek out first of all what numerical operations and their properties really are quite irrespective of any system of numerical language and notation on the bare supposition that notions are somehow formed of numbers and that names are somehow given to them.

The importance of sound principles in such a universally used instrument of education as arithmetic is so clear that it need only be hinted.

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PURE ARITHMETIC.

Article 1. Whatever can be regarded as a whole made up of parts anywise like the whole is called a *magnitude.*

A magnitude is said to be *of the same kind* as, or *of a different kind* from, another magnitude according as it can, or cannot, be thought to be either greater than, equal to, or less than, that other. For instance all straight lines are magnitudes of the same kind, and all plane surfaces are magnitudes of the same kind, but a straight line and a plane surface are magnitudes of different kinds.

Any part of a magnitude if a magnitude at all is clearly a magnitude of the same kind as the whole.

If any thing and any other thing be put together, and to the group thus made another thing be put, and again to this new group another thing be put, and so on, other groups being made successively in the same way by putting to each group when made another thing to make the next following group; and if the things which make up the several groups be viewed solely as individual members of the groups, leaving entirely out of consideration what the things are, how they are arranged in the several groups, and all else about the things:—still the groups differ from one another, and from the things contained in them, in respect of what is called the *number* of things in each of them. Accordingly such groups are spoken of as *different numbers of things*, or as *different numbers* simply; they may also be called *numerical magnitudes.*

Arithmetic is the science which deals with numerical magnitudes.

2. All particular numbers are to be regarded as defined in the following way:—Each thing in any group of things is called *one* of the things, or a *unit*, or *unity*. A group made by putting to a thing another thing is said to consist of *two* things or units. A group made by putting to a group of two things another thing is said to consist of *three* things. In like manner groups are said to consist of *four*, *five*, *six*, *seven*, &c. things, which are made respectively from groups of three, four, five, six, &c. things, by putting to them severally another thing. And all groups made up of different particular numbers of things are to be conceived in this way as made successively one from another by putting to them in order one of the things or unities. Since an endless series of such groups can be thus made it is clear

that the definitions of different specific numbers must likewise be endless. The language however which is used to express different numbers of things is so devised as in a great measure to point out the manner of their successive formation, and on this account serves in a great measure to define as well as to denote the different particular numbers referred to.

Numbers, like everything else, are considered to be equal to one another when they are perfectly alike in respect of that by which alone they are distinguished from one another. In the case of numbers indeed this equality is often regarded as amounting to sameness. A number is said to be *equal* to, or the *same* as, another number when to each unit of the one there corresponds a unit of the other,—or which is the same thing, those numbers are equal to one another which have the same name.

A number is said to be *greater* than another number when a number equal to the former is among those numbers which can be derived from the latter by putting units in succession to it and to the numbers successively obtained. The latter number in this case is also said to be *less* than the former.

3. The following principles, which may be regarded as certain elements either inherent in the fundamental notion of number or necessarily lying at the root of all numerical relations, must be postulated:—

All the units which make up any particular number are equal to one another.

Things may be numbered together as units although they should in many respects differ widely from one another. Inasmuch however as they are considered to be, all equally, separate individual things in any group of things, insomuch are they *equal* units in the number of things in the group.

Any number greater than TWO *may be made up in different ways of less numbers as parts.*

The number *one* is anything whatever. The number *two* is the only number which can be made up of nothing else than units. The number *three* may be made up of units and also of the numbers *two* and *one*. Every other number may be made up, and that in various ways, of numbers else than units as well as of units.

Other principles, admitted on all hands and by no means confined in their application to numbers, are needed besides; such as—

Whatever are equal to the same are equal to one another.

A whole is equal to the aggregate of its parts.

If equals be put to equals the wholes are equal.

A whole is greater than its part.

If equals be put to unequals the whole which contains *the greater unequal is greater than the whole which contains the less.*

A whole of which the parts are as many as and severally greater than the parts of another whole is greater than that other.

If equals be taken from equals the remainders are equal.

If equals be taken from unequals the remainder of the greater unequal is greater than the remainder of the less.

Of these last eight propositions the first four only are truly fundamental and independent of one another; and the rest in order flow out therefrom as easy consequences.

4. *To add together given numbers; that is, to find the number of which certain given numbers are the parts.*

The numbers to be added together are called *addends*, and the number of which they are the parts is called their *sum*.

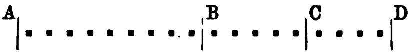
First to add together two given numbers. The addition together of every different pair of numbers constitutes the matter of a distinct proposition. If one of the pair be *one* the sum is given (art. 2) as a matter of definition. If neither be *one* the way of proceeding to find the sum is always the same and is as follows:—To add together the numbers *seven* and *six*. The aggregate of *seven* and *one* is (art. 3) equal to the sum of *seven* and *one* which by definition (art. 2) is *eight*. To each of these equals put *one* and (art. 3) the wholes are equal; therefore the aggregate of *seven, one,* and *one,* is equal to the aggregate of *eight* and *one*. But because the aggregate of *one* and *one* is (art. 2) *two,* by putting *seven* to each, the aggregate of *seven, one,* and *one,* is equal to the aggregate of *seven* and *two*. Moreover the aggregate of *eight* and *one* is equal to the sum of them or (art. 2) *nine*. Therefore the aggregate of *seven* and *two* is equal to *nine*. Again putting *one* to each of these equals it follows (art. 3) that the aggregate of *seven, two,* and *one,* is equal to the aggregate of *nine* and *one*. But because the aggregate of *two* and *one* is equal to their sum that is *three,* the aggregate of *seven, two,* and *one,* is equal to the aggregate of *seven* and *three*. Besides the aggregate of *nine* and *one* is equal to their sum that is *ten*. Therefore the aggregate of *seven* and *three* is equal to *ten*. By repeating the like process it may be shown at successive steps that the aggregate of *seven* and *four* is equal to *eleven,* of *seven* and *five* to *twelve,* and lastly of *seven* and *six* to *thirteen*. Therefore *thirteen* is the sum of *seven* and *six*.

Next to add together more than two given numbers. Let any two of the given numbers be added together in the way just shown, and let the sum so found and another of the given numbers be added together in the same way. The sum last found is the sum of the three numbers. For the aggregate of the first two numbers is (art. 3) equal to their sum, and putting the third number to each of these equals it follows (art. 3) that the aggregate of the three numbers is equal to the aggregate of the third number and the sum of the first two numbers. But the aggregate of the third number and the sum of the first two numbers is equal to the sum of that number and sum. Therefore the aggregate of the three numbers is equal to the sum of that same number and sum.

Again let the sum of the three numbers thus found and a fourth given number be added together. The new sum is the sum of the four numbers. For the new-found sum and the aggregate of the four numbers, being each equal to the aggregate of the fourth number and the sum of the first three numbers, are equal to one another.

In the same way may the sum of more than four given numbers be found.

The addition together of numbers may be represented in a palpable manner. If the numbers to be added together be represented by rows of dots



stretching from *A* to *B*, from *B* to *C*, and from

C to *D*, placed together so as to make a continuous row; the number which is the sum of these numbers will be represented by the row of dots stretching from *A* to *D*.

5. ***The final sum got by adding all but one of any given numbers in succession to that one and to the successive sums that are obtained is the same in whatever order the numbers are taken, and is the same as the sum got by adding to any one of the given numbers the sum of all the rest.***

The addition together of two numbers consists in the putting successively as many units as are contained in one of them to the other and to the numbers thus made in succession, and therefore is really not so much the addition together of the two numbers as the addition of one of them to the other. But because the aggregate of the two numbers is equal, both to the single number got by adding the second number to the first, and to the single number got by adding the first number to the second, the sums obtained in the two cases are identical.

Again, the addition together of more than two numbers consists in the adding to one of them another of them, adding to the sum so got another of them, adding to the sum then got another, and so on until the last of the numbers be reached. Now in whatever order the numbers are taken in this process the final sum is always equal to the same, viz. to the aggregate of the given numbers. Therefore the final sum is the same in whatever order the numbers are taken.

Moreover, the sum got by adding to any one of the given numbers the sum of all the rest is the same as the sum got by adding to this sum that number and therefore is the same as the final sum got by adding in succession all the given numbers except one of them taken in any order whatever to that one and to the sums that are successively obtained.

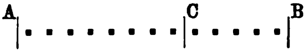
6. *From the greater of two given numbers to subtract the less; that is, having given a number and also a part of it to find the other part.*

The number to be subtracted is called the *subtrahend*, the number from which it is to be subtracted the *minuend*, and the number which together with the subtrahend makes up the minuend the *remainder*, *excess*, or *difference*.

Although the subtraction of one number from another, like the addition together of two numbers, is different for every different two numbers yet the method of proceeding is the same for any two. Thus from *seventeen* to subtract *nine*. *Seventeen* is the sum and therefore equal to the aggregate of *sixteen* and *one*. Also *sixteen* is the sum and therefore equal to the aggregate of *fifteen* and *one;* and putting *one* to each of these equals the aggregate of *sixteen* and *one* is equal to the aggregate of *fifteen*, *one*, and *one*. Therefore *seventeen* is equal to the aggregate of *fifteen*, *one*, and *one*. But because *two* is the aggregate of *one* and *one*, by putting *fifteen* to each, the aggregate of *fifteen* and *two* is equal to the aggregate of *fifteen*, *one*, and *one*. Therefore *seventeen* is equal to the aggregate of *fifteen* and *two*. Again because *fifteen* is the sum and so equal to the aggregate of *fourteen* and *one*, the aggregate of *fifteen* and *two*, and therefore also its equal *seventeen*, is equal to the aggregate of *fourteen*, *one*, and *two*. But since *three* is the sum and consequently equal to the aggregate of *one* and *two*, the aggregate of *fourteen* and *three* is equal to the aggregate of *fourteen*, *one*, and *two*. Therefore *seventeen* is equal to the aggregate of *fourteen* and *three*. In exactly the same way it may be shown at successive steps that *seventeen* is equal to the aggregate of *thirteen* and *four*, to the aggregate of *twelve* and *five*, of *eleven* and *six*, of *ten* and *seven*, and lastly of *nine* and *eight*. Hence if one part of *seventeen* be *nine* the other part is *eight*.

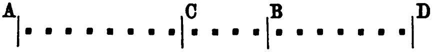
Subtraction may be done otherwise by following a course quite like that followed (art. 4) in the addition together of two numbers. Thus to subtract *sixteen* from *twenty-four*. By taking the course referred to it may be shown successively that the aggregate of *sixteen* and *one* is equal to *seventeen*, that of *sixteen* and *two* to *eighteen*, of *sixteen* and *three* to *nineteen*, of *sixteen* and *four* to *twenty*, of *sixteen* and *five* to *twenty-one*, of *sixteen* and *six* to *twenty-two*, of *sixteen* and *seven* to *twenty-three*, and of *sixteen* and *eight* to *twenty-four*. If therefore *sixteen* be subtracted from *twenty-four* the remainder is *eight*.

If from a number represented by the row of dots *AB*, a number represented by the part *CB* of that row be subtracted, the number represented by the remaining part *AC* of the row is the remainder.



7. *The remainder left after subtracting one number from another is the same as the remainder left after subtracting the sum of the former number and any number from the sum of the latter number and the selfsame number.*

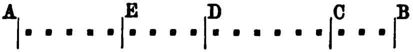
If from a number represented by the row *AB* a number represented by *CB* a part of the row be subtracted, the remainder is represented by the other part *AC*. But if to each of the numbers *AB*, *CB*, there be added any number represented by the row *BD* placed so as to be in a continuous row with



each of the rows AB, CB, the sums are respectively represented by AD, CD; and if from the number AD the number CD be subtracted the remainder is still represented by the row AC. Therefore the same remainder is got if from a number another be subtracted as if from the sum of the former number and any number the sum of the latter and the self-same number were subtracted.

8. *If numbers be subtracted in succession from a number and from the remainders successively obtained, the last remainder obtained is the same in whatever order of succession the numbers are taken, and is the same as the remainder obtained by subtracting the sum of the numbers from the number.*

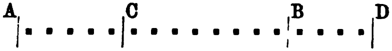
If from a number represented by AB another number represented by the part CB be subtracted, and from the remainder left represented by the other part AC another number represented by the part DC be subtracted, and from the new remainder left AD another number ED be subtracted; the last remainder AE is the same as the remainder got if from the number AB there were at once subtracted the number EB, and the number EB is the sum of the numbers CB, DC, ED. The like manifestly holds however many numbers are subtracted.



Moreover in whatever order of succession the numbers are subtracted the last remainder is always the same as the remainder got by an immediate subtraction of the sum of the numbers subtracted, and this sum is (art. 5) not changed by changing the order of the numbers. Therefore the last remainder is the same in whatever order of succession the numbers are taken.

9. *If from a number another number be subtracted and to the remainder a third number be added, the sum got is the same as the remainder which would be got if from the sum of the first and third numbers the second number were subtracted.*

If from a number represented by AB a number represented by the part AC be subtracted, and to the remainder CB a third number BD placed continuous with AB and with CB be added, the sum so got is represented by CD. But if to the number AB the number BD were added and from the sum AD the number AC were subtracted the remainder then got would also be represented by CD.



10. *The result of a series of successive operations consisting of additions and subtractions of numbers is the same in whatever possible order of succession these*

operations are performed, and is the same as the remainder got by subtracting from the sum of the number first operated on and all the numbers added the sum of all the numbers subtracted.

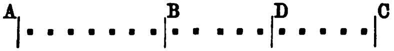
Let numbers be added in succession to a number and to the sums that are successively obtained, let numbers be then subtracted in succession from the last sum and from the remainders successively obtained, again let numbers be added in succession to the last remainder and to the sums successively obtained, again numbers subtracted in succession from the last sum and from the remainders successively obtained, other numbers added as before, others subtracted, and so on to any length. The order in which the numbers of the first set of addends are added together is (art. 5) matter of indifference; and subtracting successively the numbers of the first set of subtrahends is (art. 8) equivalent to subtracting at once the sum of these numbers; also the successive additions of the second set of addends are (art. 5) equivalent to the immediate addition of the sum of them; and so on for the other sets of subtrahends and of addends. Hence the final result is the same as would be got if from the sum of the first set of addends the sum of the first set of subtrahends were subtracted, then to the remainder the sum of the second set of addends were added, then from the sum the sum of the second set of subtrahends subtracted, and so on for the rest. Now when from the sum of the first set of addends the sum of the first set of subtrahends is subtracted and to the remainder the sum of the second set of addends is added, the result is (art. 9) the same as would be got if to the sum of the first set of addends the sum of the second set of addends were added and from the sum the sum of the first set of subtrahends were subtracted. Therefore the result got after subtracting the sum of the second set of subtrahends is the same as would be got if to the sum of the first set of addends the sum of the second set of addends were added, from the sum the sum of the first set of subtrahends were subtracted, and then from the remainder the sum of the second set of subtrahends were subtracted. But this last result again is (art. 8) the same as if from the sum, got by adding to the sum of the first set of addends the sum of the second set of addends, there were subtracted the sum got by adding to the sum of the first set of subtrahends the sum of the second set of subtrahends. Also the sum of two sums the one of one set and the other of another set of numbers is (art. 5) equal to simply the sum of all the numbers in both sets. Therefore the result got after subtracting the sum of the second set of subtrahends is the same as would be got if from the sum of the numbers in the first and second sets of addends the sum of the numbers in the first and second sets of subtrahends were subtracted. By repeating a precisely like process may it be shown that the result arrived at after adding the sum of any one of the sets of addends or after subtracting the sum of any one of the sets of subtrahends is the same as the remainder got if from the sum of all the numbers in all the sets of addends up to that set the sum of all the numbers in all the sets of subtrahends up to that set were subtracted. Hence the result of a series of successive additions and subtractions of numbers is the same as the remainder got when from the sum of the number first operated on and all the numbers added there is subtracted the sum of all the numbers subtracted.

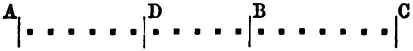
Moreover, since the sum of any numbers is (art. 5) independent of the order in which they are added together, it hence follows that the result of a series of successive additions and subtractions of numbers is the same in whatever possible order of succession the operations are performed, provided only the same numbers be used, one of them as the number first operated on and the rest of them as the numbers added, and the same numbers be used as the numbers subtracted.

Of course in any proposed order of succession of the operations the result from which a number is to be subtracted must be not less than that number; on this account particular orders of succession may be impossible.

11. *If to a number another number be added and from the sum a third number be subtracted, the remainder is the same as the result which would be got were the difference between the second and third numbers added to or subtracted from the first number according as the second number is greater or less than the third.*

To a number represented by the row of dots *AB* let there be added another number represented by the row of dots *BC* so placed as to form with *AB* an unbroken row, and from the sum represented by the whole row *AC* let a third number represented by the part *DC* be subtracted. The remainder is represented by the other part *AD*. Now if the second number be greater than the third (to which case the first figure refers) the number *AD* is the sum got by adding, to *AB* the first number, the excess *BD* of *BC* the second number over *DC* the third. But if the second number be less than the third (to which case the second figure refers) the number *AD* is the remainder got by subtracting, from *AB* the first number, the excess *DB* of *DC* the third number over *BC* the second.





12. *If to a number there be added the result of a series of successive operations consisting of additions and subtractions of numbers, the sum is the same as the result of another series of successive operations consisting of first the addition to that number of the number first operated on in the former series and then the same additions and subtractions of numbers as in the former series.*

Since the result of a series of successive additions and subtractions of numbers is (art. 10) the same as the remainder got by subtracting, from the sum of the number first operated on in the series and all the numbers added, the sum of all the numbers subtracted; the sum of any number and that result is the same as the sum of the same number and that remainder. But the sum of the number and remainder is (art. 11) the same as the result got by adding to the number the sum of the number first operated on in the series and all the numbers added and then subtracting from the sum so obtained the sum of all the numbers subtracted; and consequently is also the same as—what this result is (art. 5) the same as—the remainder got by subtracting from the sum of that number the number first operated on and all the numbers added the sum of all the numbers subtracted. Therefore the sum of a number and the result of a series of successive additions and subtractions of numbers is the same as the remainder left when from the sum of that number the number first operated on in the series and all the numbers added there is subtracted the sum of all the numbers subtracted; and therefore is (art. 10) also the same as the result of another series of successive additions and subtractions wherein any one of the numbers that make up the former sum is the number first operated on and the rest of them are the numbers added, and the numbers that make up the latter sum are the numbers subtracted.

13. *If from a number there be subtracted the result of a series of successive operations consisting of additions and subtractions of numbers, the remainder is the same as the result of another series of successive operations consisting of additions and subtractions, wherein that number and the numbers subtracted in the former series are, one of them the number first operated on and the rest of them the numbers added, and the number first operated on and the numbers added in the former series are the numbers subtracted.*

Because the result of a series of successive additions and subtractions is (art. 10) the same as the remainder got by subtracting from the sum of the number first operated on in the series and all the numbers added the sum of all the numbers subtracted, any number exceeds that result by the same as it exceeds that remainder. But the excess of a number over that remainder is (art. 11) the same as the excess of the sum got by adding to the number the sum of the numbers subtracted in the series of additions and subtractions over the sum of the number first operated on in the series and all the numbers added, which again is (art. 5) the same as the excess of the sum of that number and all the numbers subtracted in the series over the sum of the number first operated on in the series and all the numbers added. Therefore the remainder got by subtracting from a number the result of a series of successive additions and subtractions of numbers is the same as the remainder got by subtracting from the sum of that number and all the numbers subtracted in the series the sum of the number first operated on in the series and all the numbers added, and is therefore (art. 10) also the same as the result of another series of successive additions and subtractions in which the first-mentioned number and the numbers subtracted in the former series are taken one of them for the number first operated on and the rest of them for the numbers added, and the number first operated on in the former series and the numbers added are taken for the numbers subtracted.

14. *To multiply one given number by another; that is, to find the number which is made up of as many equal parts each equal to the former given number as are expressed by the latter.*

The number to be multiplied is called the *multiplicand,* the number which expresses how many parts each equal to the multiplicand are to make up the whole the *multiplier,* and the number which is the whole so made up the *product.*

The product of the multiplication of one number by another is precisely the sum of as many numbers each equal to the multiplicand as are expressed by the multiplier and therefore may be found by the method of article 4.

15. *Equimultiples of equals are equal.*

A magnitude wholly made up of parts each equal to another magnitude is called a *multiple* of that other.

If equals be put to equals the wholes are (art. 3) equal, and if equals be put to the equal wholes the new wholes are equal, and so on. Hence a whole of which the parts are as many as, and severally equal to, the parts of another whole is equal to that other. If moreover all the parts of each of the wholes be equal to the same, it follows that equimultiples of equals are equal.

In particular if the equals be equal numbers, the products made by multiplying equals by equals are equal.

16. *If a number be multiplied by another number, and the product thus made be multiplied by another*

number, and the new product by another, and so on; the final product is the same as the product made by multiplying the first number by the final product got by multiplying by the third, fourth, fifth, &c. numbers in succession the second number and the products successively obtained.

Let groups of things be made each containing as many things as are expressed by the first of any given numbers; then let groups of these groups be made each containing as many of them as are expressed by the second of the given numbers; again let groups of a third class be made each containing as many groups of the second class as are expressed by the third of the given numbers; and so on, each group of any class being made to contain as many groups of the immediately preceding class as are expressed by the corresponding given number. The number of things in each group of the second class is expressed by the product of the multiplication of the first number by the second. Therefore the number of things in each group of the third class is expressed by the product of the multiplication of that product by the third number. And so on for the rest. Therefore at last the number of things in each group of the last-made class is expressed by the final product got by multiplying by the second, third, fourth, &c. numbers in succession the first number and the products successively obtained. But in like manner it follows that the number of groups of the first class contained in each group of the last-made class is expressed by the final product got by multiplying by the third, fourth, fifth, &c. numbers in succession the second number and the products successively obtained. Consequently the number of things in each group of the last-made class is expressed by the product made by multiplying the first of the given numbers by that final product. Since then the number of things in each group of the last-made class is expressed, both by the final product got by multiplying by the second, third, fourth, &c. numbers in succession the first number and the successive products obtained, and by the product of the multiplication of the first number by the final product got by multiplying by the third, fourth, fifth, &c. numbers in succession the second number and the successive products obtained; the results of these two different sets of operations must be identical.

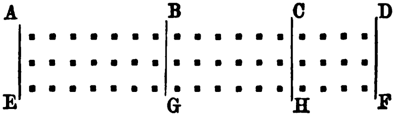
17. *The product made by multiplying one number by another is the same as the product made by multiplying the second number by the first.*

Let things be arranged in a set of groups, each group containing as many things as are expressed by any number, and the set containing as many groups as are expressed by any other number. The number of things in the set of groups is expressed by the product made by multiplying the first number by the second. Let the same things be again arranged in a new set of groups by taking one thing from each group of the old set to form each group of the new. Each new group contains as many things as are expressed by the second number, and the new set contains as many groups as are expressed by the first number. Therefore the number of things is expressed by the product made by multiplying the second number by the first. Hence the same number is got for the product of the multiplication of one number by another and for the product of the multiplication of the second number by the first.

The common result of these two different operations may be spoken of as the product of the multiplication together of the two numbers or shortly the product of the two numbers.

18. *The product made by multiplying together a number and the sum of other numbers is the same as the sum of the products made by multiplying together that number and each of the other numbers.*

Let any numbers be represented severally by rows of dots AB, BC, CD, placed in an unbroken row so that their sum is represented by the whole row AD. Let as many such whole rows and their parts be placed side by side as are expressed by any number, so as to form the whole system of dots $AEFD$ and the parts of that system $AEGB$, $BGHC$, $CHFD$. The system $AEFD$ represents the product made by multiplying together the number and the sum of the numbers AB, BC, CD, and represents also the sum of the numbers represented severally by the systems $AEGB$, $BGHC$, $CHFD$. But these last mentioned numbers are the products made by multiplying together the number and each of the numbers AB, BC, CD. Hence the product of a number and the sum of numbers is the same as the sum of the products of the number and each of the numbers.



19. *If the first two in any set of numbers be multiplied together, and the product thus made and the next number in the set be multiplied together, and the new product made and the next number then multiplied together, and so on to the last number in the set; the final product obtained is the same in whatever order the numbers are arranged in the set.*

Let two numbers be multiplied together and let the product made be multiplied by a third number. Because the product made by multiplying together the first and second numbers is (arts. 14 and 17) equal both to the sum of as many numbers each equal to the first number as are expressed by the second number and to the sum of as many numbers each equal to the second number as are expressed by the first number, the product made by multiplying that product by the third number is equal to each of the products made by multiplying each of those sums by the third number, and therefore is (art. 18) equal both to the sum of as many numbers each equal to the product made by multiplying together the first and third numbers as are expressed by the second number and to the sum of as many numbers each equal to the product made by multiplying together the second and third numbers as are expressed by the first number. But the former of these sums is (art. 14) the product made by multiplying the product of the first and third numbers by the second number, and the latter is the product made by multiplying the product of the second and third numbers by the first number. Therefore each of these products is the same as the product made by multiplying the product of the first and second numbers by the third number.

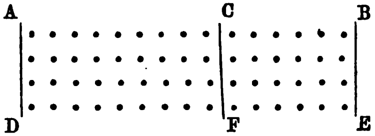
Again let the common result of these various operations, which may be called the product of the multiplication together of three numbers, be multiplied by a fourth number. Since, by what has just been proved, the product made by multiplying together the first, second, and third numbers is equal, firstly to the sum of as many numbers each equal to the product of the second and third numbers as are expressed by the first number, secondly to the sum of as many numbers each equal to the product of the first and third numbers as are expressed by the second number, and thirdly to the sum of as many numbers each equal to the product of the first and second numbers as are expressed by the third number; the product made by multiplying that product by the fourth number is equal to each of the products made by multiplying those sums severally by the fourth number, and consequently is (art. 18) also equal, firstly to the sum of as many numbers each equal to the product of the second, third, and fourth numbers as are expressed by the first number, secondly to the sum of as many numbers each equal to the product of the first, third, and fourth numbers as are expressed by the second number, and thirdly to the sum of as many numbers each equal to the product of the first, second, and fourth numbers as are expressed by the third number. But these sums are severally (art. 14) firstly the product made by multiplying by the first number the product of the multiplication together of the second, third, and fourth numbers, secondly the product made by multiplying by the second number the product of the multiplication together of the first, third, and fourth numbers, and thirdly the product made by multiplying by the third number the product of the multiplication together of the first, second, and fourth numbers. Therefore each of these products is equal to the product made by multiplying by the fourth number the product of the multiplication together of the first, second, and third numbers.

In like manner it may be shown that the product made by multiplying by a fifth number the product of the multiplication together of four numbers is equal to each of the products made by multiplying by each of these four numbers the product of the multiplication together of the other three and the fifth number. And by repeating the like process it may be shown for any number of numbers that if two of the numbers be multiplied together, and the product made and another of the numbers be multiplied together, and the product then made and another of the numbers, and so on until the numbers be exhausted, the last product made is the same in whatever order the numbers are taken.

Numbers when thus multiplied together are called the *factors* of the product made.

20. *The product made by multiplying together a number and the difference of two other numbers is the same as the difference of the products made by multiplying together that number and each of the two other numbers.*

Let the row of dots *AB* represent the greater of two numbers and the part *CB* of that row the less, so that the remaining part *AC* represents the difference. Let as many such rows and parts of rows as are expressed by any third number be placed side by side thus forming the whole system of dots *ADEB* and the parts of that system *ADFC, CFEB.* The systems *ADEB, CFEB,* represent the products made by multiplying together the third number and each of the two numbers, and the system *ADFC* represents the difference of these products and also the product made by multiplying together the third number and the difference of the two numbers. Therefore the product of a number and the difference of two numbers is the same as the difference of the products of that number and each of the two.



21. *The product made by multiplying together a number and the result of a series of successive additions and subtractions of numbers is the same as the result of the like series of successive additions and subtractions of the products made by multiplying together the first number and each of the other numbers.*

The product made by multiplying together a number and the result of a series of successive additions and subtractions of numbers, being (art. 10) the same as the product made by multiplying together that number and the difference of two sums the one of the number first operated on and all the numbers added in the series and the other of all the numbers subtracted, is (art. 20) the same as the difference of the products made by multiplying together that number and each of those sums, and therefore is (art. 18) also the same as the difference of two sums the one of the products made by multiplying together that number and each of the numbers yielding the former sum and the other of the products made by multiplying together that same number and each of the numbers yielding the latter sum. But this last-mentioned difference is (art. 10) the same as the result of a series of successive additions and subtractions in which the number first operated on and the numbers added are the products that make up the former of the two last-mentioned sums and the numbers subtracted are the products that make up the latter. Therefore the product made by multiplying together a number and the result of a series of successive additions and subtractions of other numbers is the same as the result of the like series of successive additions and subtractions of the products made by multiplying together that number and each of the other numbers.

22. *To divide one of two given numbers by the other; that is, to find the number of equal parts each equal to the latter given number which either by themselves or together with a part less than that number make up the former given number as a whole.*

The number to be divided is called the *dividend*, the number to which each of the equal parts of the dividend is to be equal the *divisor*, the number which expresses how many of these equal parts the dividend contains the *quotient*, and the remaining part of the dividend the *remainder* of the division.

If from the dividend a number equal to the divisor be subtracted, and from the remainder left another precisely equal number be subtracted, and from the remainder then left another and so on, until at length there be left either nothing at all or a remainder less than the divisor; the number expressing how many of these numbers have been so subtracted is the quotient, and the remainder of the last subtraction is the remainder of the division. For the last remainder is (art. 8) the same as the remainder which would be got by at once subtracting the sum of these numbers.

The product made by multiplying the divisor by the quotient either is equal to the dividend or is less than and differs from the dividend by a number less than the divisor.

When the dividend is wholly made up of parts each equal to the divisor the division is called *exact*, in other words an exact division is one in which there is no remainder. For a reason which will appear farther on, all the divisions spoken of in the following articles are none other than exact divisions; so that in every case that follows the dividend is equal to the sum of as many numbers each equal to the divisor as are expressed by the quotient, or which is the same thing is equal to the product of the multiplication by the quotient of the divisor. In such case the quotient expresses the multiple (art. 15) which the dividend is of the divisor.

23. *If a magnitude be a multiple of another magnitude any magnitude equal to the first is the same multiple of any magnitude equal to the second.*

If equals be taken from equals the remainders are (art. 3) equal, and if the like equals be taken from the equal remainders the new remainders are equal, and so on. Hence if at any stage one of the original equals be exhausted so likewise is the other.

In particular when the equals are equal numbers, the quotients got by dividing equals by equals are equal.

24. *If the division of one number by another give a third number for quotient, the division of the first number by the third gives the second number for quotient.*

If the division of one number by another give a third number for quotient, the first number is (art. 22) equal to the product made by multiplying the second number by the third, and therefore is (art. 17) also equal to the product made by multiplying the third number by the second. Hence if the first number be divided by the third number the quotient obtained is (art. 22) identical with the second number.

25. *The quotient got by dividing one number by another is the same as the quotient got by dividing the product of the multiplication by the former number of any number by the product of the multiplication by the latter number of that same number.*

Since in any exact division the dividend is equal to the product made by multiplying the divisor by the quotient, the product made by multiplying any number by the dividend is equal to the product made by multiplying that same number by the product of the multiplication of the divisor by the quotient, and therefore is (art. 16) also equal to the product of the multiplication by the quotient of the product made by multiplying the said same number by the divisor. Hence if the product made by multiplying any number by the dividend be divided by the product made by multiplying the same number by the divisor the quotient obtained is (art. 22) identical with the quotient of the original division.

26. *If a number be divided by another number, and the quotient got be divided by another number, and the quotient then got by another, and so on; the final quotient is the same in whatever order of succession the divisions by the numbers are performed, and is the same as the quotient got by dividing the first number by the product made by multiplying together all the other numbers.*

In a series of divisions by numbers in succession of a number and of the quotients that are successively obtained, the first dividend is (art. 22) equal to the product made by multiplying the first divisor by the first quotient; also the first quotient, being the second dividend, is equal to the product made by multiplying the second divisor by the second quotient. Therefore the first dividend is equal to the product made by multiplying the first divisor by the product of the multiplication by the second quotient of the second divisor, and therefore is (art. 16) also equal to the product of the multiplication by the second quotient of the product made by multiplying the first divisor by the second. Again, because the second quotient in the series of divisions is the third dividend and consequently equal to the product made by multiplying the third divisor by the third quotient, it follows in exactly the same way that the first dividend is equal to the product of the multiplication by the *third quotient* of the final product made by multiplying the first divisor by the second and the product so made by the third divisor. And by successive precisely like steps it at last follows that the first dividend is equal to the product of the multiplication by the last quotient of the final product made by multiplying the first divisor by the second, the product thus made by the third, the product then made by the fourth, and so on to the last of the divisors. Therefore the last quotient in the series of divisions is (art. 22) identical with the quotient got by dividing the first dividend by the final product made by multiplying by the second, third, fourth, &c. divisors in succession the first divisor and the products successively made.

Now if the same numbers be used as divisors but the order of succession of the divisions by them be changed in any way, the last quotients obtained are always the same as the quotients got by dividing the first dividend by the final products made by multiplying by all the divisors except one of them in succession that one and the products successively made. But all these final products are (art. 19) equal to one another. Hence if there be divided by given numbers in succession a given number and the quotients that are successively obtained, the final quotient is the same in whatever order of succession the divisions by the given numbers are performed.

and is the same as the quotient got by dividing the given dividend by the product of the multiplication together of all the given divisors.

27. *If a number be divided by another number and the quotient be multiplied by a third number, the product is the same as the quotient of the division by the second number of the product made by multiplying the first number by the third.*

Since when one number is divided by another the first is (art. 22) equal to the product made by multiplying the second by the quotient, the product made by multiplying the first number by any third number is (art. 15) equal to the final product made by multiplying the second number by the quotient of the division of the first number by the second and then multiplying the product so made by the third number, and is consequently (art. 16) also equal to the product made by multiplying the second number by the product of the multiplication by the third number of the quotient got by dividing the first number by the second. Therefore if the product made by multiplying the first number by the third be divided by the second, the quotient obtained is (art. 22) the same as the product made by multiplying by the third number the quotient of the division of the first by the second.

28. *The result of a series of successive operations consisting of multiplications and divisions by numbers is the same in whatever possible order of succession these operations are performed, and is the same as the quotient got by dividing the product of the multiplication together of the number first operated on and all the numbers used as multipliers by the product of the multiplication together of all the numbers used as divisors.*

Let there be multiplied by numbers in succession a number and the products that are successively obtained, let there be then divided by numbers in succession the last product and the quotients that are successively obtained, again let there be multiplied by numbers in succession the last quotient and the products that are successively obtained, again let numbers be used as before for successive divisors, then numbers for successive multipliers, and so on to any length. Since the successive divisions by the numbers in any set of divisors give (art. 26) the same result as an immediate division by the product of the numbers in the set, and the successive multiplications by the numbers in any set of multipliers give (arts. 16 and 19) the same result as an immediate multiplication by the product of the numbers in the set, the result of the series of successive multiplications and divisions is the same as the result of another series of successive operations which consists in dividing the product of the number first operated on and all the numbers in the first set of multipliers by the product of all the numbers in the first set of divisors, multiplying the quotient thus got by the product of all the numbers in the second set of multipliers, dividing the product then got by the product of all the numbers in the second set of divisors, and so on to the end of the sets of multipliers and divisors. Now in the latter series of operations, because a division by one number followed by a multiplication of the quotient by another gives (art. 27) the same result as a multiplication by the latter number followed by a division of the product by the former, the result after the second division is the same as would be got by multiplying the product of the number first operated on and the numbers in the first set of multipliers by the product of the numbers in the second set of multipliers, dividing the product so got by the product of the numbers in the first set of divisors, and dividing the quotient then got by the product of the numbers in the second set of divisors. Again these last-mentioned two successive divisions give (art. 26) the same result as an immediate division by the product of the two divisors; and the product of two products is (arts. 16 and 19) the same as the product of all the factors of both products. Hence the result got at the end of the *second set of* divisions in the original series of operations, or after the second division in the equivalent series, is the same as the quotient got by dividing the product of the multiplication together of the number first operated on and all the numbers in the first and second sets of multipliers by the product of the multiplication together of all the numbers in the first and second sets of divisors. In the same way may it be shown that the result got at the end of the third set of divisions in the original series of operations or after the third division in the equivalent series is the same as the quotient arising from the division of the product of the number first operated on and all the numbers in the first, second, and third sets of multipliers by the product of all the numbers in the first, second, and third sets of divisors. And by successive precisely like steps it at last follows that the final result of a series of successive multiplications and divisions is the same as the quotient got from the division of the product made by multiplying together the number first operated on and all the numbers used as multipliers by the product made by multiplying together all the numbers used as divisors.

Moreover, since this quotient is (art. 19) always the same provided only the same numbers be used one of them as the number first operated on and the rest as multipliers and the same numbers be used as divisors, it follows that the order of succession of a series of successive multiplications and divisions by given numbers nowise affects the final result.

In multiplications and divisions by given numbers in succession of a given number and the results successively obtained there may be particular orders of succession of the operations in which an inexact division is proposed to be effected. The proposition must for the present (art. 22) be supposed to apply to such orders of succession only as give rise to exact divisions.

29. *The sum of the quotients got by dividing each of any numbers by one and the same number is the same as the quotient got by dividing the sum of those numbers by that number.*

If each of any numbers be divided by another number, the sum of those numbers is (art. 22) equal to the sum of the products made by multiplying the common divisor by each of the resulting quotients; and consequently is (art. 18) also equal to the product made by multiplying the common divisor by the sum of the quotients. Hence if the sum of the dividends be divided by the common divisor the quotient obtained is (art. 22) identical with the sum of the quotients.

30 *The difference of the quotients got by dividing each of two numbers by the same number is the same*

as the quotient got by dividing the difference of the two numbers by that same number.

If each of two numbers be divided by a third number, the difference of the two numbers is (art. 22) equal to the difference of the products made by multiplying the third number by each of the quotients; and therefore is (art. 20) also equal to the product made by multiplying the third number by the difference of the quotients. Hence if the difference of the two numbers be divided by the third number the quotient got is (art. 22) the same as the difference of the quotients.

31. *The result of a series of successive additions and subtractions of the quotients arising from the divisions of each of any numbers by one and the same number is the same as the quotient arising from the division by the self-same number of the result of the like series of successive additions and subtractions of the numbers themselves.*

If each of any numbers be divided by one and the same number, the result of a series of successive additions and subtractions of the numbers is (art. 22) equal to the result of the like series of successive additions and subtractions of the corresponding products made by multiplying the common divisor by each of the quotients; and hence is (art. 21) also equal to the product made by multiplying the common divisor by the result of the exactly like series of successive additions and subtractions of the corresponding quotients. Therefore if the result of the series of successive additions and subtractions of the numbers be divided by the common divisor, the quotient got is (art. 22) the same as the result of the like series of successive additions and subtractions of the quotients got by dividing the numbers severally by the common divisor.

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32. If a magnitude be wholly broken up into any number of parts all of which are equal to one another, each of these parts or whatever is equal to each of them is called a *measure* or *submultiple* of the magnitude; and any one or any group either of these parts or of whatever magnitudes are equal to them constitutes what is called a *fraction* of the magnitude. In other words a fraction of a magnitude is simply any multiple (art. 15) of any submultiple of the magnitude.

In order to fix on firm ground the following explanations and definitions a few simple matters concerning submultiples must be noticed. If a magnitude be wholly broken up into any number of equal parts and an equal magnitude be likewise wholly broken up into the same number of equal parts, each of the parts of the one is equal to each of the parts of the other, or more shortly equisubmultiples of equals are equal. For otherwise, each of the parts of one of the magnitudes would be greater than each of the parts of the other, and a magnitude of which the parts are as many as and severally greater than the parts of another magnitude would be not greater than that other, which is (art. 3) impossible. Again any submultiple of the greater of two unequal magnitudes is greater than the same submultiple of the less. For the submultiple of the greater can be neither equal to nor less than the submultiple of the less; since in the former case equimultiples of equals would be not equal, and in the latter a whole of which the parts are as many as and severally greater than the parts of another whole would be less and therefore not greater than that other, and both cases are (arts. 15 and 3) impossible. Hence once more if a magnitude be wholly broken up first into a certain number of equal parts and then into a greater number of equal parts, each of the latter parts is less than each of the former. Because so many of the latter parts as there are of the former make up only a portion of the magnitude, and consequently each of the former parts stands to each of the latter in the relation of a submultiple of the greater of two unequal magnitudes to the same submultiple of the less.

Since a magnitude may be wholly broken up into equal parts in endless different ways there are endless different ranks or classes of submultiples of any one magnitude. Any particular class of submultiples of a magnitude may however be readily distinguished from all other classes of submultiples of the same magnitude by simply noting how many submultiples of the class make up the magnitude. Accordingly the number which expresses how many submultiples of a certain class or denomination make up the magnitude is called the *denominator* of all fractions of the magnitude that consist of such submultiples. The number which expresses how many submultiples of the class indicated by the denominator are contained in any fraction of a magnitude is called the *numerator* of the fraction. Fractions are named by means of their numerators and denominators as in the following instances:—If a magnitude be wholly broken up into *seven* equal parts, a magnitude consisting of *four* of these parts or submultiples is called *four seventh-parts* or simply *four-sevenths* of the first magnitude, another magnitude made up of *twelve* of such submultiples *twelve-sevenths;* the first of these magnitudes is likewise *seven-fourths* of the second magnitude and *seven-twelfths* of the third; moreover the second magnitude is *four-twelfths* of the *third, and the* third *twelve-fourths* of the second.

The numerator and denominator of a fraction are called its *terms*.

Any fraction of a magnitude which contains fewer submultiples of the magnitude than the magnitude itself is called a *proper* fraction of the magnitude; any other fraction of a magnitude is called an *improper* fraction of it. Any improper fraction of a magnitude is manifestly either a number of magnitudes each equal to that magnitude (viz. when the numerator of the fraction is exactly divisible by the denominator) or a number of such magnitudes and a proper fraction of one of them taken together (viz. when the numerator is not exactly divisible by the denominator). On the other hand any number of equal magnitudes may be expressed as an improper fraction of one of them having any given number for denominator; for, each of the magnitudes being broken up into as many equal parts as are expressed by the given number, the magnitudes consist of as many of these equal parts as are expressed by the product made by multiplying that given number by the number expressing how many magnitudes there are, and hence are the fraction of one of themselves whose numerator is this product and denominator the given number. It is immediately evident that a magnitude may be expressed as an improper fraction of itself by simply taking any number for both numerator and denominator.

Thus it appears that every magnitude and that every group of equal magnitudes is actually a fraction. Hence all the possible different fractions of any magnitude include among them all the possible different numbers of whatever magnitudes are equal to that magnitude; and consequently single magnitudes, groups of equal magnitudes, and what are only fractions of magnitudes, may be all considered together under one and the same general notion. Instead of introducing a new word for this notion, the common practice in language, of making the word already used for a portion of a class stand for the whole, is followed; and the term *number*, which at first was only applied to any thing taken singly and to any group of things, now receives a wider meaning and is further applied to any fraction of a thing—provided of course the thing be a magnitude and so capable of furnishing fractions of itself. Numbers of whatever magnitudes are equal to one another, whether single magnitudes, groups of magnitudes, or only fractions of magnitudes are all said to refer to the *same unit* or *unit-magnitude.* When it is necessary to distinguish single units and groups of units from what are merely fractions of units the former are called *whole* numbers and the latter *fractional* numbers.

Fractions of any unit-magnitude are actually whole *numbers of certain* new classes or orders of units, and these new units are precisely the various submultiples of the unit-magnitude. With a view to distinguish the subordinate units of these new classes from the primary units to which fractions refer, and at the same time to mark that they are submultiples of those units, they may be called the *subunits* of which the fractions are made up. Units of different orders are not here considered for the first time; since in the operation of multiplication the multiplier refers to units each equal to the number of primary units expressed by the multiplicand, and in the operation of division the quotient refers to units each equal to the number of primary units expressed by the divisor. The units here introduced however are submultiples of the primary unit whereas the units to which the multiplier and quotient have reference are multiples of the primary unit. Whatever the primary unit be there can always be multiples of it, but there can be submultiples of the primary unit only when it is a magnitude. It is always understood that units are magnitudes unless it be expressly stated that they are not.

33. *A fraction is equal to another fraction of which the terms are the products made by multiplying any whole number by the corresponding terms of the former.*

If each of the subunits that make up any fraction, and also each of the subunits of the same denomination that make up the unit to which the fraction refers, be broken up into as many equal parts as are expressed by any whole number; all these parts are equal to one another, because equisubmultiples of equal subunits are equal; and hence each of them is at once a submultiple of the fraction and a submultiple of the unit to which the fraction refers. Further, the fraction contains so many of them as are expressed by the product made by multiplying the whole number by the numerator of the fraction, and the unit so many as are expressed by the product made by multiplying the whole number by the denominator. Therefore the fraction is equal to another fraction of which the former product is the numerator and the latter product the denominator. In fact the fraction thus found is only the original fraction expressed in other terms.

34. *Any fractions being given having all different denominators, to find fractions severally equal to them having all the same denominator.*

Since each of any fractions that have all different denominators is (art. 33) equal to a fraction whose terms are the products made by multiplying any whole number by the corresponding terms of that fraction, it is only required to find as many whole numbers as there are fractions which when multiplied severally *by the denominators* shall all produce the same number. If a whole number can be found which is exactly divisible by each of the denominators the quotients got from dividing this whole number by the several denominators fulfil the condition required; for the product made by multiplying by any denominator the quotient of the division of the whole number by that denominator is (art. 17) the same as the product made by multiplying the denominator by the quotient and therefore is (art. 22) the same as the whole number itself. Endless whole numbers can be found that are exactly divisible by every one of the denominators. In particular, the product of the multiplication together of all the denominators is such a whole number, being (arts. 16 and 19) identical with the product made by multiplying any one of the denominators by the product of the multiplication together of all the rest.

The matter of this article may be stated in other words:—Having given fractions expressed in subunits of different denominations to express the same fractions in subunits of one and the same denomination.

35. *To add together given fractions, and from the greater of two given fractions to subtract the less.*

To add together given fractions is to find the fraction of which the given fractions are the parts. The given fractions, when they have a common denominator are, and when they have not a common denominator may (art. 34) be expressed as, whole numbers of subunits of the same denomination, and consequently are to be added together in the same way as any whole numbers (art. 4). Therefore the sum is a fraction having for numerator the sum of the numerators, either of the fractions themselves when they have a common denominator, or of fractions severally equal to them that have a common denominator when themselves have not, and for denominator the common denominator, either of the fractions, or of the fractions equal to the fractions.

From the greater of two given fractions to subtract the less is to find the fraction which together with the latter given fraction makes up the former. The given fractions either are, or may be expressed as, whole numbers of the same class of subunits, and therefore their difference is a fraction whose numerator is the difference (art. 6) of the numerators, either of the fractions themselves when they have, or of fractions respectively equal to them that have, when themselves have not, a common denominator, and whose denominator is the common denominator, either of, or of the fractions equal to, the fractions.

36. Whatever sets of operations, consisting of *additions only, of subtractions only, or of additions and subtractions only, are equivalent to one another in the case of whole numbers are also equivalent to one another in the case of fractional numbers.*

The sets of operations referred to are those stated at length and shown to be equivalent to one another for whole numbers in arts. 5, 7—13; and the headings of these articles taken together are a fit heading for the present article provided numbers fractional as well as whole be understood whenever numbers are mentioned in them.

The additions and subtractions of fractions are really only the additions and subtractions of whole numbers of the same class of subunits, of that class to wit which is indicated by the common denominator, either of the fractions themselves, or of fractions severally equal to them. Therefore whatever is true of additions and subtractions of whole numbers in general is true of the additions and subtractions of these whole numbers in particular.

37. *To explain the nature of the operations called* MULTIPLICATION *and* DIVISION *of fractions; and to show how they are performed.*

To multiply a fraction by a whole number needs no explanation; according to the meaning already given to multiplication (art. 14) it is simply to find the sum of as many fractions each equal to the fractional multiplicand as are expressed by the whole multiplier. This may be done by the method of art. 35, and hence the product is a fraction whose numerator is the product made by multiplying the numerator of the multiplicand by the multiplier and whose denominator is the denominator of the multiplicand.

A new operation now comes to be considered, viz., the operation of finding the fraction which is a certain given fraction of another given fraction. Since all whole numbers are (art. 32) actually fractions, this is clearly a general operation which includes as a particular case the operation of finding the whole number which is made up of a given whole number of whole numbers each equal to a given whole number. Accordingly the term *multiplication*, which hitherto has been applied only to the latter and particular operation, is henceforth applied also to the former and general operation. The extension of meaning here given to the term *multiplication* is akin to, and indeed arises immediately out of, that before given (art. 32) to the term *number*.

To multiply then one given fraction by another, that is, according to the explanation just given, to find the fraction which is such fraction of the former given fraction (or multiplicand) as is ex*pressed by the* latter given fraction (or multiplier).

There must manifestly be found first that submultiple of the multiplicand which is indicated by the denominator of the multiplier and then that multiple of the submultiple thus got which is expressed by the numerator of the multiplier. Let each of the subunits that make up the multiplicand, and likewise each of the subunits of the same rank that make up the unit, be broken up into as many equal parts as are expressed by the denominator of the multiplier; all these parts, being equisubmultiples of equals, are equal to one another; each of them is consequently the submultiple of the unit indicated by the product made by multiplying the denominator of the multiplier by the denominator of the multiplicand; and moreover there are in the multiplicand as many groups of them as are expressed by the numerator of the multiplicand each of which contains as many of them as are expressed by the denominator of the multiplier. Next let the equal parts into which the multiplicand has been thus wholly broken up be arranged anew in other groups by taking one of the parts from each of the old groups in order to form each of the new; the new groups so made, being equimultiples of equals, are all equal to one another; each new group contains as many of the parts as there were old groups, that is, as many as are expressed by the numerator of the multiplicand; and there are as many new groups as there were parts in each of the old groups, that is, as many as are expressed by the denominator of the multiplier. Therefore the submultiple of the multiplicand indicated by the denominator of the multiplier is just whatever is equal to one of these new groups, that is, a fraction having for numerator the numerator of the multiplicand and for denominator the product made by multiplying the denominator of the multiplier by the denominator of the multiplicand. Whence, by simply multiplying this fraction by the numerator of the multiplier, it at once follows that the product made by multiplying one fraction by another is a fraction whose numerator is the product made by multiplying the numerator of the multiplicand by the numerator of the multiplier and whose denominator is the product made by multiplying the denominator of the multiplier by the denominator of the multiplicand.

The like result may be otherwise reached. Were the denominator of the multiplier the same as the numerator of the multiplicand, each of the subunits which make up the multiplicand would be that submultiple of the multiplicand which is indicated by the denominator of the multiplier, and so many of them as are expressed by the numerator of the multiplier would be that fraction of the multiplicand which is expressed by the multiplier. In such a case therefore *the product* would be a fraction having for numerator the numerator of the multiplier and for denominator the denominator of the multiplicand. Now when the denominator of the multiplier is not the same as the numerator of the multiplicand, fractions may be found the one equal to the multiplier and the other to the multiplicand such that the denominator of the former shall be the same as the numerator of the latter, and these fractions, being in fact merely other expressions for, may be used instead of, the multiplier and multiplicand. In any case the multiplier is (art. 33) equal to a fraction whose terms are the products made by multiplying the numerator of the multiplicand by the corresponding terms of the multiplier, and the multiplicand is equal to a fraction whose terms are the products made by multiplying the denominator of the multiplier by the corresponding terms of the multiplicand; moreover the denominator of the fraction so got equal to the multiplier is (art. 17) the same as the numerator of the fraction so got equal to the multiplicand. Hence as before the product is a fraction having for numerator the product made by multiplying the numerator of the multiplicand by the numerator of the multiplier and for denominator the product made by multiplying the denominator of the multiplier by the denominator of the multiplicand.

When the numerator of the multiplicand is exactly divisible by the denominator of the multiplier the product may be found more simply. The division being effected the subunits that make up the multiplicand may be disposed in as many groups as are expressed by the quotient each containing as many of the subunits as are expressed by the denominator of the multiplier; and by taking one subunit out of each of these groups a new group will be formed which is that submultiple of the multiplicand indicated by the denominator of the multiplier. For by taking another subunit out of each of the groups another precisely equal new group would be formed, and by taking another out of each another precisely equal new group would be formed, and so on until at length and at the same time all the groups would be exhausted, viz., when from each had been taken as many subunits as are expressed by the denominator of the multiplier. Whence, by taking the multiple expressed by the numerator of the multiplier, it follows that the product is a fraction whose numerator is the product made by multiplying the quotient by the numerator of the multiplier and whose denominator is the denominator of the multiplicand. The same result may be got by expressing the multiplier in terms of the products of the multiplications by its terms of the quotient; since the product made by multiplying the quotient by the denominator of the multiplier, being (art. 17) the same as that made by *multiplying the* denominator of the multiplier by the quotient, is (art. 22) the same as the numerator of the multiplicand. The result here got is readily seen (arts. 33 and 16) to agree with the result got by applying the general method to this case.

The operation of finding the fraction which is a given fraction of a given fraction immediately suggests another operation, that of finding the fraction which one given fraction is of another given fraction. It is at once manifest that this is a general operation which includes as a special case the operation of finding the whole number that expresses the multiple which one given whole number is of another given whole number. Accordingly, taking the same course as before, the name *division* already bestowed on the particular operation only is now bestowed on the general operation likewise. In fact if a single magnitude or group of equal magnitudes and a fraction of a magnitude be classed together and both called *numbers*; and if the operation of finding the whole number which is made up of a given whole number of given equal whole numbers and the operation of finding the fraction which is a given fraction of a given fraction be classed together and both called *multiplicative operations*; it seems to be required for the sake of mere consistency that the operation of finding the whole number of given equal whole numbers which *produce a given* whole number and the operation of finding the fraction of a given fraction which produces a given fraction be classed together and both called *divisive* operations.

Let it be required to divide one given fraction by another, that is, as has just been explained, to find the fraction which the former given fraction (or dividend) is of the latter given fraction (or divisor). If the dividend and divisor have the same denominator they are the whole numbers expressed by their respective numerators of subunits of the same rank, and consequently the dividend is that fraction of the divisor whose numerator is the numerator of the dividend and whose denominator is the numerator of the divisor. If the dividend and divisor have not the same denominator, fractions may (art. 34) be found that are respectively equal to them and that have the same denominator, and as before the dividend is that fraction of the divisor whose numerator is the numerator of the fraction equal to the dividend and whose denominator is the numerator of the fraction equal to the divisor. In any case the dividend is equal to a fraction whose terms are the products made by multiplying the denominator of the divisor by the corresponding terms of the dividend, and the divisor is equal to a fraction whose terms are the products made by multiplying the denominator of the dividend by the corresponding terms of the divisor; also these *fractions equal* respectively to the dividend and divisor have (art. 17) the same denominator. Hence in any case the quotient is a fraction having for numerator the product made by multiplying the denominator of the divisor by the numerator of the dividend and for denominator the product made by multiplying the denominator of the dividend by the numerator of the divisor.

The division of the unit by a fraction deserves separate notice. Because so many of the subunits that make up any fraction as are expressed by the denominator make up the unit to which the fraction refers, the fraction which unity is of any fraction, or in other words, the quotient of the division of unity by any fraction, is a fraction having for numerator the denominator and for denominator the numerator of that fraction. Again if unity be divided by a fraction expressed in the same terms as the quotient so got, the quotient now got is a fraction expressed in the same terms as the original fraction. When two fractions are in this way so related that each expresses the fraction which unity is of the other, or that the fraction of each expressed by the other is unity, each is called the *reciprocal* of the other.

By using the property of the reciprocal the division of one fraction by another may readily be done. For, the dividend being the fraction of the unit which itself expresses and the unit being the fraction of the divisor which the reciprocal of the divisor expresses, it follows at once that the dividend is the fraction of the divisor expressed by the product made by multiplying the reciprocal of the divisor by the dividend. This manifestly tallies with the result before got.

From the very nature of a fraction it is plain that one whole number is that fraction of another whole number which has for numerator the former whole number and for denominator the latter; this then is the quotient of the division of the one by the other. On comparing this quotient with that got by the method of art. 22 it is seen that when the dividend is exactly divisible by the divisor the results (art. 33) agree; but that otherwise the quotient here got exceeds the other by the fraction which the remainder of an inexact division is of the divisor. The fact is that when the dividend is not exactly divisible by the divisor the operation of art. 22 is not a division (as the term is now understood) of the whole dividend but only a division of that part of the dividend which together with the remainder makes up the whole. Whence it appears that divisions performed by the method of art. 22 can be regarded as complete operations only in the case of exact divisions, for in the case of inexact divisions account cannot be taken in the quotient of the remainder and so far there fails to be effected a complete division or measurement of the dividend by the divisor. Any inexact division may *however be rendered* complete by simply adding to the incomplete quotient the fraction which the remainder is of the divisor, and the sum is the complete quotient. Throughout arts. 23—31 all the divisions spoken of were supposed to be exact because such are the only complete divisions that can be effected without using fractions. Whenever operations of division are mentioned complete divisions are meant unless incomplete divisions are expressly specified.

Here it may be noticed that in any multiplicative operation the product expresses in reference to the chief unit that which the multiplier expresses in reference to the multiplicand as unit, and in any divisive operation the quotient expresses in reference to the divisor as unit that which the dividend expresses in reference to the chief unit. Wherefore in multiplication the multiplicand and product both refer to the unit-in-chief while the multiplier refers to the multiplicand as unit, and in division the dividend and divisor both refer to the unit-in-chief while the quotient refers to the divisor as unit.

38. In order to complete so far as is at present possible the system of extended notions of number and numerical operations which have been seen to spring out of the notion of a fraction, an extension may now be given to this notion itself. It is readily suggested that as a magnitude may be estimated in

reference to another magnitude by means of the whole numbers of parts each equal to a third magnitude of which the first and second magnitudes are respectively made up, so may a magnitude be estimated in reference to another magnitude by means of the fractions which these magnitudes are respectively of a third magnitude. Whence arises the notion of what may be called a fraction having a fraction for numerator and a fraction for denominator. For the sake of distinction a fraction whose terms are whole numbers is called a *simple* fraction, and a fraction whose terms are fractional numbers a *complex* fraction.

A complex fraction of a magnitude may be expressed in the form of a simple fraction. For a complex fraction is the simple fraction expressed by the fractional numerator of such a magnitude as is determined by the condition that the simple fraction of it expressed by the fractional denominator is the unit-magnitude to which the complex fraction refers. If then the fractional terms have the same denominator, the complex fraction consists of the whole number, expressed by the numerator of the fractional numerator, of those submultiples of the said magnitude that are indicated by the common denominator, and the unit-magnitude consists of the whole number of those same submultiples expressed by the numerator of the fractional denominator; therefore the complex fraction is identical with the simple fraction having for numerator the numerator of the numerator and for denominator the numerator of the denominator. If again the fractional terms have not the same denominator, there are fractions respectively equal to them that have the same denominator which being only other expressions for them may be used in their stead. By taking instead of the numerator the fraction whose terms are the products made by multiplying the denominator of the denominator by the corresponding terms of the numerator, and instead of the denominator the fraction whose terms are the products made by multiplying the denominator of the numerator by the corresponding terms of the denominator, the complex fraction is expressed in fractional terms that have (art. 17) the same denominator. Hence in any case a complex fraction is identical with a simple fraction whose numerator is the product made by multiplying the denominator of the denominator by the numerator of the numerator and whose denominator is the product made by multiplying the denominator of the numerator by the numerator of the denominator.

Since a complex fraction denotes the fraction expressed by its numerator of that whereof the fraction expressed by its denominator is the unit, the quotient of the division of one simple fraction by another may manifestly be at once expressed in the form of a complex fraction. The quotient is the complex fraction whose numerator is the dividend and whose denominator is the divisor. By expressing this complex fraction in the form of a simple fraction the result is seen perfectly to agree with foregoing results. Moreover, because that whereof a certain simple fraction is the unit is expressed by the simple fraction's reciprocal, a complex fraction is the same as the simple fraction which is the product made by multiplying the reciprocal of the denominator by the numerator.

In the same way as the notion of a complex fraction is got from the notion of a simple fraction by taking simple fractions instead of whole numbers for the terms, so may a notion be got of a still more complex fraction by taking complex fractions instead of simple fractions for the terms. Such a fraction, of the second degree of complexity, may be expressed in the form of a simple fraction by first expressing each of its terms in the form of a simple fraction in the way just shown and then expressing in the same way the fraction of the first degree of complexity so got in the form of a simple fraction.

So likewise may notions be got of fractions of higher and higher degrees of complexity by taking fractions of any degree of complexity for the terms of a fraction of the next higher degree. Also a fraction *of any degree of* complexity may be expressed step by step in the forms of fractions of lower and lower degrees and at last in the form of a simple fraction.

It is clear that complex fractions of whatever degrees of complexity may enter into additions subtractions multiplications and divisions and that these operations may be performed by the methods already given since any fraction however complex may be expressed in the form of a simple fraction.

39. *If equals be multiplied by equals the products are equal. And if equals be divided by equals the quotients are equal.*

If equals be multiplied by equals the equal multipliers either are, or may (arts. 34, 38) be expressed as, simple fractions that have the same denominator. And these fractions must also have the same numerator, for otherwise they would not be equal to one another. Now the equisubmultiples of the equal multiplicands indicated by the common denominator of these fractions are (art. 32) equal, and therefore the equimultiples of those submultiples expressed by the common numerator are (art. 15) equal. Hence the products made by multiplying equals by equals are equal.

It is likewise clear that if equals be multiplied by unequals the product of the multiplication by the greater unequal is greater than the product of the multiplication by the less. For as before the unequal multipliers either are or may be expressed as simple fractions that have the same denominator, and the greater of these fractions must have the greater numerator. Hence the product made by multiplying by the greater multiplier exceeds the product made by multiplying by the less by the multiple expressed by the excess of the greater numerator over the less of the submultiple of either of the equal multiplicands indicated by the common denominator.

From this it follows at once that the quotients of the divisions of equals by equals are equal. For the products made by multiplying the equal divisors by the quotients being equal to the equal dividends are equal to one another, and from what has just been shown this cannot be unless the quotients are equal.

40. *Whatever sets of operations are equivalent to one another in the case of whole numbers are also equivalent to one another in the case of fractional numbers.*

The sets of operations here referred to are those stated at length and proved to be equivalent to one another in the case of whole numbers throughout arts. 5, 7—13, 16—21, 24—31, and the headings of these articles taken altogether will serve as a fit heading for this article if whenever numbers are *mentioned* numbers fractional as well as whole be understood. So far as additions and subtractions only are concerned the proposition has already (art. 36) been proved.

In so far as multiplications, either alone or combined with additions and subtractions only, are concerned, the equivalences stated and proved for whole numbers throughout arts. 16—21 are readily seen to depend wholly on the following four propositions:—

(1) The final product made by multiplying by the second third fourth &c. of any numbers in succession the first of them and the products successively made is the same as the product made by multiplying the first number by the final product made by multiplying by the third fourth fifth &c. numbers in succession the second number and the products successively made.

(2) If two of any set of numbers be multiplied together, and the product so made and another number of the set be multiplied together, then this new product and another of the set, and so on to the end of the set, the final product is the same in whatever order the numbers of the set are taken.

(3) The product made by multiplying together a number and the sum of other numbers is the same as the sum of the products made by multiplying together that number and each of the other numbers.

(4) The product made by multiplying together a number and the difference of two numbers is the same as the difference of the products made by multiplying together that number and each of the two numbers.

Hence these being proved for fractional numbers the proposition itself becomes proved so far as additions subtractions and multiplications only are concerned.

(1) Let there be any given fractions. Let as many magnitudes as there are fractions be taken such that the first of them is the fraction of the unit-magnitude expressed by the first of the given fractions, the second magnitude the fraction of the first expressed by the second of the given fractions, the third magnitude the fraction of the second expressed by the third of the fractions, and so on for the rest, each magnitude being the fraction of the magnitude next before it expressed by the corresponding given fraction. The second magnitude is the fraction of the unit-magnitude expressed by the product made by multiplying the first fraction by the second. Hence the third magnitude is the fraction of the unit expressed by the final product made by multiplying the first fraction by the second and then multiplying the product so made by the third fraction. Hence by successive precisely like steps it follows that the last *of the magnitudes* is the fraction of the unit-magnitude

expressed by the final product made by multiplying by the second third fourth &c. fractions in succession the first fraction and the products successively made. Now in exactly the same way it appears that the last of the magnitudes is the fraction of the first of them expressed by the final product made by multiplying by the third fourth fifth &c. of the fractions in succession the second of them and the products successively made; and therefore the last of the magnitudes is the fraction of the unit-magnitude expressed by the product made by multiplying the first fraction by the last-mentioned final product. These two expressions of the last of the magnitudes in reference to the unit-magnitude are consequently identical.

(2) The product made by multiplying one fraction by another is (art. 37) a fraction whose numerator is the product made by multiplying the numerator of the former by the numerator of the latter and whose denominator is the product made by multiplying the denominator of the latter by the denominator of the former. But (art. 17) each of these products is the same as the product made by interchanging the multiplicand and multiplier. Therefore the product made by multiplying one fraction by another is the same as the product made by multiplying the latter by the former; and hence it may be said that the *product made* by multiplying together two fractions

is a fraction whose numerator is the product made by multiplying together their numerators and whose denominator is the product made by multiplying together their denominators. Again if the product of the two fractions and a third fraction be multiplied together the product made is in like manner a fraction whose terms are the products made by multiplying together the corresponding terms of the third fraction and of the product of the multiplication together of the two fractions. But neither of these products is (art. 19) at all affected by changing the order of the factors. Therefore the same final product is obtained however the order of the three fractions be changed; and this final product, which may be called simply the product made by multiplying together the three fractions, is a fraction whose numerator is the product of the multiplication together of the three numerators and whose denominator is the product of the multiplication together of the three denominators. In exactly the same way it follows (arts. 37, 17, and 19) that if any two out of a set of fractions be multiplied together, and the product so made and another fraction of the set be multiplied together, and the product then made and another of the set, and so on until the whole set be exhausted, the final product is a fraction whose numerator is the product made by multiplying together all the numerators of the fractions and whose *denominator is* the product made by multiplying together all the denominators, and consequently is the same in whatever order the fractions are taken out of the set.

(3) Since the sum of any fractions is (art. 35) a fraction whose numerator is the sum of the numerators, either of the fractions themselves when they have all the same denominator, or of fractions severally equal to them that have all the same denominator when the fractions themselves have not all the same denominator, and whose denominator is the common denominator either of the fractions themselves or of the fractions severally equal to them; the product made by multiplying together any other fraction and that sum is a fraction having for numerator the product of the multiplication together of the numerator of the fraction and the forementioned sum of numerators and for denominator the product of the multiplication together of the denominator of the fraction and the forementioned common denominator. But the fraction thus got is (art. 18) identical with a fraction having the same denominator but having for numerator the sum of the products made by multiplying together the numerator of the single fraction and each of the forementioned numerators, and this fraction again is (art. 35) the sum of the products made by multiplying together the single fraction and each of the fractions whose sum was found. Therefore the product made by multiplying together a fraction

and the sum of fractions is the same as the sum of the products made by multiplying together the fraction and each of the fractions.

(4) Because the difference of two fractions is (art. 35) a fraction having for numerator the difference of the numerators, either of the fractions themselves when they have a common denominator, or of fractions respectively equal to them that have a common denominator when themselves have not, and for denominator the common denominator either of the fractions or of the fractions respectively equal to them; the product made by multiplying together a third fraction and the difference of the two fractions is a fraction whose numerator is the product made by multiplying together the numerator of the third fraction and the difference of the numerators aforesaid and whose denominator is the product made by multiplying together the denominator of the third fraction and the common denominator aforesaid. But the difference of the products made by multiplying together the third fraction and each of the two fractions is (art. 35) a fraction whose numerator is the difference of the products made by multiplying together the numerator of the third fraction and each of the aforesaid numerators and whose denominator is the product made by multiplying together the denominator of the third fraction and the aforesaid common denominator. Now the fractions got by

these different processes are identical since they have manifestly the same denominator and have (art. 20) also the same numerator. Therefore the product made by multiplying together a fraction and the difference of two fractions is the same as the difference of the products made by multiplying together the first and each of the two.

Lastly, the proposition being thus established so far as additions subtractions and multiplications one two or all are concerned, it is easily extended to divisions also. For all the conclusions arrived at throughout arts. 24—31 (the articles relating to sets of operations in which divisions enter) depend entirely on previous conclusions and on the single fact that in any complete division the dividend is equal to the product made by multiplying the divisor by the quotient. Hence all that is said throughout these articles applies word for word to the case of fractions, numbers being now understood to mean numbers fractional as well as whole.

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